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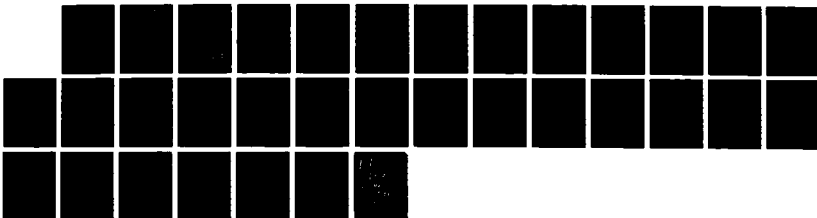
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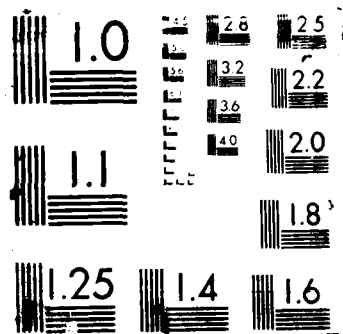
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MEASUREMENTS: A CONTINUOUS DEPENDENCE RESULT

By

Avner Friedman

and

Michael Vogelius

IMA Preprint Series # 373

December 1987

INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS

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IDENTIFICATION OF SMALL INHOMOGENEITIES OF EXTREME CONDUCTIVITY BY BOUNDARY MEASUREMENTS: A CONTINUOUS DEPENDENCE RESULT

AVNER FRIEDMAN[†] AND MICHAEL VOGELIUS[‡]

Abstract. We consider an electrostatic problem for a conductor consisting of finitely many small inhomogeneities of extreme conductivity, embedded in a spatially varying reference medium. Firstly we establish an asymptotic formula for the voltage potential in terms of the reference voltage potential, the location of the inhomogeneities and their geometry. Secondly we use this representation formula to prove a Lipschitz continuous dependence estimate for the corresponding inverse problem. This estimate bounds the difference in the location and in the relative size of two sets of inhomogeneities by the difference in the boundary voltage potentials corresponding to a fixed current distribution.

§1 Introduction and statement of the main result. The determination of conductivity profiles from knowledge of boundary currents and voltages has recently received a lot of attention in the literature. In the biomedical community a common name appears to have emerged for such work: *electrical impedance imaging* [6]. It is usually assumed that the (direct current) voltage potential u satisfies the differential equation

$$\nabla \cdot (\gamma(x) \nabla u) = 0 \quad \text{in } \Omega, \quad \Omega \subset \mathbf{R}^n, \quad n \geq 2$$

where $\gamma(x)$ is the positive, real valued conductivity, to be determined. The extra information, based upon which it is sought to determine $\gamma(x)$, consists of knowledge of currents $\gamma \frac{\partial u}{\partial \nu}$ and the corresponding voltage potentials u at the boundary, $\partial\Omega$.

Let Λ_γ denote the linear operator from $H^{1/2}(\partial\Omega)$ into $H^{-1/2}(\partial\Omega)$ which takes Dirichlet- to Neumann-data:

$$\Lambda_\gamma(\phi) = \gamma \frac{\partial u}{\partial \nu} \quad \text{with } \nabla \cdot (\gamma \nabla u) = 0 \quad \text{in } \Omega, u = \phi \text{ on } \partial\Omega.$$

Complete knowledge of Λ_γ is known to determine the function γ uniquely under quite general assumptions: This was verified for analytic and piecewise analytic γ in [11] and [12], for C^∞ γ and dimension $n \geq 3$ in [18], and under the assumption that γ be sufficiently close to a constant it was verified for C^∞ γ and dimension $n = 2$ in [17]. For a layered conductor, i.e., $\gamma = \gamma(x_1)$, two sets of Dirichlet- and Neumann-data suffice to determine even a bounded measurable γ (cf. [13]).

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Because of inevitable error in measurements, it is at least as important to study the continuity of the dependence of γ on Λ_γ as it is to verify uniqueness. Alessandrini [2] has recently examined this question for $n \geq 3$ and shown that the mapping $\Lambda_\gamma \rightarrow \gamma$ has a modulus of continuity of logarithmic type, provided γ is a priori known to belong to a bounded set in some Sobolev space; his proof exploits the continuity of the mappings $\Lambda_\gamma \rightarrow \gamma|_{\partial\Omega}$ and $\Lambda_\gamma \rightarrow \partial\gamma/\partial\nu|_{\partial\Omega}$ established in [19]. The assertion of Alessandrini is, more specifically, that if Λ_{γ_1} and Λ_{γ_2} deviate by δ (in the operator norm on $B(H^{1/2}, H^{-1/2})$) then γ_1 and γ_2 deviate at most by $C(\log \frac{1}{\delta})^{-\sigma}$ (in $L^\infty(\Omega)$) for some $0 < \sigma < 1$. While such a result is theoretically very interesting, it is at the same time somewhat disappointing, since it predicts quite a weak form of continuous dependence. It does not explain the apparent practical success of various numerical algorithms to recover γ from only partial knowledge of Λ_γ (cf. [4], [14], [20] and [21]). To bridge the gap it seems relevant to analyze the continuous dependence for interesting classes of conductivities, where the functional dependence on x is further restricted; one example of such analysis is found in [5], [9]. It is of practical importance to seek continuous dependence estimates in terms of only finitely many sets of Dirichlet- and Neumann- data.

In this paper we consider conductivities that correspond to a finite number of small inhomogeneities, with extreme conductivity, imbedded in an n -dimensional reference medium. The reference conductivity $\gamma(x)$ satisfies

$$(1.1) \quad 0 < c_0 \leq \gamma(x) \leq C_0 < \infty.$$

We assume that each inhomogeneity has the form $z_k + \epsilon\rho_k B$ where B is some bounded domain in \mathbf{R}^n with

$$(1.2) \quad 0 \in B \text{ and } \partial B \text{ of type } C^{2+\beta}, \text{ for some } 0 < \beta < 1$$

The points $\{z_k\}_{k=1}^K$ belong to Ω and satisfy:

$$(1.3) \quad |z_k - z_j| \geq d_0 > 0, \quad \forall j \neq k, \quad \text{and} \\ \text{dist}(z_k, \partial\Omega) \geq d_0 > 0, \quad \forall k.$$

The parameter ϵ determines the common length scale of the inhomogeneities, and the ρ_k .

$$(1.4) \quad d_0 \leq \rho_k \leq D_0,$$

determine their relative size. We always assume that ϵ is small enough that the sets $z_k + 2\epsilon\rho_k B$ are disjoint and that their distance to $\mathbf{R}^n \setminus \Omega$ is larger than $\frac{d_0}{2}$.

Let

$$\omega_\epsilon = \bigcup_{k=1}^K (z_k + \epsilon\rho_k B)$$

denote the total collection of inhomogeneities. If they all have infinite conductivity, then the voltage potential $u_\epsilon = u_\epsilon^\infty$, given the boundary current ψ , is the solution to

$$(1.5) \quad \min_{u \in H^1(\Omega), \nabla u = 0 \text{ in } \omega_\epsilon} \left\{ \frac{1}{2} \int_{\Omega} \gamma |\nabla u|^2 dx - \int_{\partial\Omega} \psi u ds \right\}.$$

If all the inhomogeneities have conductivity 0, then the voltage potential $u_\epsilon = u_\epsilon^0$ solves

$$(1.6) \quad \min_{u \in H^1(\Omega \setminus \omega_\epsilon)} \left\{ \frac{1}{2} \int_{\Omega \setminus \omega_\epsilon} \gamma |\nabla u|^2 dx - \int_{\partial\Omega} \psi u ds \right\}.$$

We assume that

$$(1.7) \quad \Omega \text{ is bounded with } \partial\Omega \in C^{2+\beta}.$$

REMARK 1.1. For problems (1.5), (1.6) to have a solution it is necessary and sufficient that $\psi \in H^{-1/2}(\partial\Omega)$ with

$$(1.8) \quad \int_{\partial\Omega} \psi ds = 0.$$

We shall make the solution $u = u_\epsilon$ of (1.5), or (1.6), unique by requiring that

$$(1.9) \quad \int_{\partial\Omega} u_\epsilon ds = 0.$$

The solutions u_ϵ^∞ and u_ϵ^0 may be obtained as limits from an electrostatic problem involving finite and nonzero conductivity: Let $\gamma_\alpha = \gamma 1_{\Omega \setminus \omega_\epsilon} + \alpha 1_{\omega_\epsilon}$, $\alpha > 0$, and denote by u_ϵ^α the solution of

$$\min_{u \in H^1(\Omega)} \left\{ \frac{1}{2} \int_{\Omega} \gamma_\alpha |\nabla u|^2 dx - \int_{\partial\Omega} \psi u ds \right\}.$$

Then

$$(1.11) \quad u_\epsilon^\infty = \lim_{\alpha \rightarrow \infty} u_\epsilon^\alpha \text{ and } u_\epsilon^0 = \lim_{\alpha \rightarrow 0} u_\epsilon^\alpha,$$

the first limit is relative to $H^1(\Omega)$, the second relative to $H^1(\Omega \setminus \omega_\epsilon)$.

In most of this paper we shall focus our attention on the problem with inhomogeneities of infinite conductivity; the case of zero conductivity is very similar and is treated briefly in the final section.



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We shall henceforth write u_ϵ instead of u_ϵ^∞ . Our main tool of analysis is to express u_ϵ in terms of U , the solution of

$$(1.12) \quad \min_{u \in H^1(\Omega)} \left\{ \frac{1}{2} \int_{\Omega} \gamma |\nabla u|^2 - \int_{\partial\Omega} \psi u ds \right\} ;$$

U is the voltage potential corresponding to the reference medium alone, and it is normalized by

$$(1.13) \quad \int_{\partial\Omega} U ds = 0$$

We assume that

$$(1.14) \quad \gamma \in C^{2+\beta}(\bar{\Omega}),$$

and that the following *non-degeneracy condition* holds:

$$(1.15) \quad \nabla U(x) \neq 0 \quad \forall x \in \Omega .$$

In practice this means that we can choose any γ -harmonic function U , with $\nabla U(x) \neq 0$ in Ω , and then apply the boundary current $\psi = \gamma \partial U / \partial \nu$ on $\partial\Omega$.

Consider two arbitrary collections of inhomogeneities

$$\omega_\epsilon = \bigcup_{k=1}^K (z_k + \epsilon \rho_k B) \text{ and } \omega'_\epsilon = \bigcup_{k=1}^{K'} (z'_k + \epsilon \rho'_k B)$$

both satisfying (1.3), (1.4) and denote by u_ϵ and u'_ϵ the corresponding voltage potentials (with fixed boundary current, ψ). Our main result is the following continuous dependence theorem:

THEOREM 1.1. *Let Γ be a given nonempty open subset of $\partial\Omega$. There exist constants $0 < \epsilon_0$, δ_0 and C and a function $\eta(\epsilon)$, $\lim_{\epsilon \rightarrow 0} \eta(\epsilon) = 0$, such that if $\epsilon < \epsilon_0$ and $\epsilon^{-n} \|u_\epsilon - u'_\epsilon\|_{L^\infty(\Gamma)} < \delta_0$ then*

$$(i) \quad K = K', \text{ and, after appropriate reordering,}$$

$$(ii) \quad |z_k - z'_k| + |\rho_k - \rho'_k|$$

$$\leq C \epsilon^{-n} \|u_\epsilon - u'_\epsilon\|_{L^\infty(\Gamma)} + \eta(\epsilon) \quad (1 \leq k \leq K) .$$

The constants ϵ_0, δ_0 and C and the function η depend on $d_0, D_0, \Gamma, \Omega, \gamma$ and ψ but are otherwise independent of the two sets of inhomogeneities.

The factor ϵ^{-n} in front of $\|u_\epsilon - u'_\epsilon\|_{L^\infty(\Gamma)}$ is best possible; it follows immediately from our analysis (cf. Lemma 3.3) that even if $|z_k - z'_k|$ and $|\rho_k - \rho'_k|$ is of order 1 then the discrepancy in the boundary data is of order ϵ^n . For fixed (and small) ϵ our theorem basically shows that the locations of the inhomogeneities, z_k , and their relative sizes, ρ_k , depend Lipschitz-continuously on the rescaled boundary deviation $\epsilon^{-n} \|u_\epsilon - u'_\epsilon\|_{L^\infty(\Gamma)}$.

In the formulation of the theorem we used the L^∞ norm of $(u_\epsilon - u'_\epsilon)|_\Gamma$; this is not essential, in fact the analysis in Section 4 shows that other norms can be used, such as the L^1 norm.

A brief outline of this paper is as follows. Theorem 1.1 is proved in section 4. The proof is based on an asymptotic representation formula for u_ϵ , as derived in section 3. To establish this representation formula we require some energy estimates of $U - u_\epsilon$; these estimates are found in section 2.

REMARK 1.2. Recently Friedman [8] has shown that the presence of an inhomogeneity or a collection of inhomogeneities can be detected by boundary measurement of the voltage potential (on Γ) corresponding to a single current distribution. For the case of small inhomogeneities of extreme conductivity Theorem 1.1 goes much further. It shows that the exact locations and the relative sizes are determined, in a continuous fashion, by the same measurement.

REMARK 1.3. Theorem 1.1 can be extended to the case where each inhomogeneity is of the form $z_k + \epsilon \rho_k B_k$ and B_k are different, fixed domains with smooth boundaries.

§2 Energy estimates. In this section we estimate the difference between u_ϵ and the reference potential U . The first result concerns the $H^1(\Omega)$ -norm of $U - u_\epsilon$.

LEMMA 2.1. *There exists a constant C such that*

$$\int_{\Omega} (|\nabla(U - u_\epsilon)|^2 + |U - u_\epsilon|^2) dx \leq C \epsilon^n \|v\|_{H^{-1/2}(\partial\Omega)}^2.$$

Proof. Since

$$\int_{\Omega} |v|^2 dx \leq C \left(\int_{\Omega} |\nabla v|^2 dx + \left| \int_{\partial\Omega} v ds \right|^2 \right)$$

and $\int_{\partial\Omega} (U - u_\epsilon) ds = 0$, it suffices to prove that

$$(2.1) \quad \int_{\Omega} |\nabla(U - u_{\epsilon})|^2 dx \leq C \epsilon^n \|\psi\|_{H^{-1/2}(\partial\Omega)}^2 .$$

Set

$$V_{\epsilon} = \{v \in H^1(\Omega) : \nabla v = 0 \text{ in } \omega_{\epsilon}, \int_{\partial\Omega} v ds = 0\} .$$

From the definitions of u^{ϵ}, U we get

$$(2.2) \quad \int_{\Omega} \gamma \nabla u_{\epsilon} \cdot \nabla w dx = \int_{\partial\Omega} \psi w ds = \int_{\Omega} \nabla U \cdot \nabla w dx \quad \forall w \in V_{\epsilon} ,$$

which implies that u_{ϵ} is the projection of U onto the space V_{ϵ} . Therefore

$$(2.3) \quad \int_{\Omega} \gamma |\nabla(U - u_{\epsilon})|^2 dx = \min_{v \in V_{\epsilon}} \int_{\Omega} \gamma |\nabla(U - v)|^2 dx .$$

Since γ is bounded from above and away from 0, it suffices to show that there exists a $v_{\epsilon} \in V_{\epsilon}$ such that

$$(2.4) \quad \int_{\Omega} |\nabla(U - v_{\epsilon})|^2 dx \leq C \epsilon^n \|\psi\|_{H^{-1/2}(\partial\Omega)}^2 .$$

Consider first the case of one inhomogeneity (with $z = 0$ and $\rho = 1$). Let

$$c_{\epsilon} = \frac{1}{(2\epsilon)^n |B|} \int_{2\epsilon B} U dx .$$

A rescaling of the Poincaré inequality (on $2B$) gives that

$$(2.5) \quad \int_{2\epsilon B} |U - c_{\epsilon}|^2 dx \leq C \epsilon^2 \int_{2\epsilon B} |\nabla U|^2 dx .$$

Define

$$(2.6) \quad w_{\epsilon}(x) = \phi\left(\frac{x}{\epsilon}\right) c_{\epsilon} + (1 - \phi\left(\frac{x}{\epsilon}\right)) U ,$$

where ϕ is a C^1 cutoff function:

$$(2.7) \quad \phi(y) = \begin{cases} 1 & \text{for } y \in B \\ 0 & \text{for } y \in \mathbb{R}^n \setminus 2B . \end{cases}$$

Clearly

$$v_\epsilon = w_\epsilon - \frac{1}{|\partial\Omega|} \int_{\partial\Omega} w_\epsilon \, ds$$

is an element of V_ϵ . From (2.6), (2.7) it follows that

$$\begin{aligned} \int_{\Omega} |\nabla(U - v_\epsilon)|^2 \, dx &= \int_{\Omega} |\nabla(U - w_\epsilon)|^2 \, dx \\ &\leq \int_{\epsilon B} |\nabla U|^2 \, dx + \int_{2\epsilon B \setminus \epsilon B} |\nabla(U - w_\epsilon)|^2 \, dx, \end{aligned}$$

and thus, using (2.5), (2.6), we get

$$(2.8) \quad \int_{\Omega} |\nabla(U - v_\epsilon)|^2 \, dx \leq C \int_{2\epsilon B} |\nabla U|^2 \, dx.$$

By elliptic estimates

$$\int_{2\epsilon B} |\nabla U|^2 \, dx \leq |B| 2^n \epsilon^n \|\nabla U\|_{L^\infty(2\epsilon B)} \leq C \epsilon^n \|\psi\|_{H^{-1/2}(\partial\Omega)},$$

and in combination with (2.8) this proves (2.4).

So far we have considered one inhomogeneity. In the case of K inhomogeneities the result follows from the previous proof and a localization argument. \square

Lemma 2.1 asserts that $\|U - u_\epsilon\|_{H^1(\Omega)} = O(\epsilon^{n/2})$; from the trace theorem it therefore follows that $\|U - u_\epsilon\|_{H^{1/2}(\partial\Omega)} = O(\epsilon^{n/2})$. This however is not the best possible estimate, in fact we have:

LEMMA 2.2. *There exists a constant C such that*

$$\|U - u_\epsilon\|_{H^{1/2}(\partial\Omega)} \leq C \epsilon^n \{1 + \|\psi\|_{H^{-1/2}(\partial\Omega)}^2\}.$$

Proof. Let W and w_ϵ be solutions to the same minimization problems as U and u_ϵ , just with ψ replaced by $\tilde{\psi}$. From the Schwarz inequality and Lemma 2.1 we get that

$$(2.9) \quad \left| \int_{\Omega} \gamma \nabla(W - w_\epsilon) \cdot \nabla(U - u_\epsilon) \, dx \right| \leq C \epsilon^n (\|\psi\|_{H^{-1/2}(\partial\Omega)}^2 + \|\tilde{\psi}\|_{H^{-1/2}(\partial\Omega)}^2).$$

Since $w_\epsilon \in V_\epsilon$, it follows from (2.2) that

$$(2.10) \quad \int_{\Omega} \gamma \nabla w_\epsilon \cdot \nabla(U - u_\epsilon) \, dx = 0.$$

Integration by parts yields

$$(2.11) \quad \int_{\Omega} \gamma \nabla W \cdot \nabla (U - u_{\epsilon}) \, dx = \int_{\partial\Omega} (U - u_{\epsilon}) \tilde{\psi} \, ds ,$$

since $\nabla \cdot (\gamma \nabla W) = 0$ in Ω and $\gamma \frac{\partial W}{\partial \nu} = \tilde{\psi}$ on $\partial\Omega$. A combination of (2.9), (2.10) and (2.11) gives

$$\left| \int_{\partial\Omega} (U - u_{\epsilon}) \tilde{\psi} \, dx \right| \leq C \epsilon^n (\|\psi\|_{H^{-1/2}(\partial\Omega)}^2 + \|\tilde{\psi}\|_{H^{-1/2}(\partial\Omega)}^2) ,$$

and thus by taking the maximum over $\tilde{\psi}$ subject to $\|\tilde{\psi}\|_{H^{-1/2}(\partial\Omega)} \leq 1$ and $\int_{\partial\Omega} \tilde{\psi} \, ds = 0$, we get the assertion of the lemma. \square

REMARK 2.1. The estimate in Lemma 2.2 follows directly from the representation formula for u_{ϵ} which we shall derive in the next section. Indeed, it follows that $U - u_{\epsilon}$ is of order ϵ^n , near $\partial\Omega$, in the $C^{1+\beta'}$ -norm, $0 < \beta' < \beta$; however the present proof is simpler and we have included it for completeness.

§3 A representation formula. We now proceed to find an asymptotic expression for u_{ϵ} . This expression will involve the functions Φ_j ($j = 1, \dots, n$) which solve the exterior problems

$$(3.1) \quad \begin{aligned} \Delta \Phi_j &= 0 && \text{in } \mathbf{R}^n \setminus \overline{B} , \\ \Phi_j &= -y_j && \text{on } \partial B , \\ \Phi_j &\text{ is uniformly bounded in } \mathbf{R}^n \setminus B \text{ and} \\ \text{if } n \geq 3 : \Phi_j(y) &\rightarrow 0 && \text{as } |y| \rightarrow \infty ; \end{aligned}$$

such Φ_j exist and are unique, [15]. For $n \geq 3$ we shall also need the function ϕ , which satisfies

$$(3.2) \quad \begin{aligned} \Delta \phi &= 0 && \text{in } \mathbf{R}^n \setminus \overline{B} , \\ \phi &= 1 && \text{on } \partial B , \\ \phi(y) &\rightarrow 0 && \text{as } |y| \rightarrow \infty . \end{aligned}$$

The following lemma is well known, and is only stated here for the convenience of the reader.

LEMMA 3.1. *Let $\Phi(y)$ be a uniformly bounded harmonic function in $\mathbf{R}^n \setminus \overline{B}$. If $n \geq 3$, assume that $\Phi(y) \rightarrow 0$ as $|y| \rightarrow \infty$. Then there exists some constant c such that*

$$\begin{aligned} \Phi(y) &= c |y|^{2-n} + o(|y|^{1-n}), \text{ and} \\ \nabla \Phi(y) &= \nabla(c |y|^{2-n}) + o(|y|^{-n}) \quad \text{as } |y| \rightarrow \infty. \end{aligned}$$

Briefly the lemma follows by considering the Kelvin transform

$$\Phi'(y') = |y'|^{2-n} \Phi\left(\frac{y'}{|y'|^2}\right), \quad y' \text{ near } 0,$$

and noting that the singularity of Φ' at $y' = 0$ is removable since

$$\Phi'(y') = \begin{cases} o\left(\log \frac{1}{|y'|}\right) & \text{if } n = 2 \\ o(|y'|^{2-n}) & \text{if } n \geq 3. \end{cases}$$

From Lemma 3.1 it follows that

$$\int_{\{|y|=R\}} \left(\Phi_i \frac{\partial \Phi_j}{\partial \nu} - \Phi_j \frac{\partial \Phi_i}{\partial \nu} \right) ds \rightarrow 0 \quad \text{as } R \rightarrow \infty,$$

and Green's formula therefore yields

$$(3.3) \quad \int_{\partial B} \Phi_i \frac{\partial \Phi_j}{\partial \nu} ds = \int_{\partial B} \Phi_j \frac{\partial \Phi_i}{\partial \nu} ds.$$

Similarly,

$$(3.4) \quad \int_{\partial B} \frac{\partial \phi}{\partial \nu} y_i ds = - \int_{\partial B} \phi \frac{\partial \Phi_i}{\partial \nu} ds = - \int_{\partial B} \frac{\partial \Phi_i}{\partial \nu} ds, \quad n \geq 3.$$

Notice, by the maximum principle, that $0 < \phi < 1$ in $\mathbf{R}^n \setminus \overline{B}$ and

$$(3.5) \quad -\frac{\partial \phi}{\partial \nu} > 0 \quad \text{on } \partial B$$

(ν is the outward normal relative to B).

We now prove two crucial estimates, needed for establishing the representation formula.

LEMMA 3.2. *For any $\psi \in H^{-1/2}(\partial\Omega)$ there exists a constant C such that*

$$(3.6) \quad \int_{\partial \omega_\epsilon} \left| \frac{\partial u_\epsilon}{\partial \nu} \right| ds \leq C \epsilon^{n-1}, \quad \text{and}$$

$$(3.7) \quad \int_{\partial(z_k + \epsilon \rho_k B)} \gamma \frac{\partial u_\epsilon}{\partial \nu}(x - z_k) ds_x = (\epsilon \rho_k)^n \gamma(z_k) A \nabla U(z_k) + o(\epsilon^n)$$

for $1 \leq k \leq K$, uniformly in $\{z_k\}$ and $\{\rho_k\}$. $A = (a_{ij})$ is the symmetric positive definite matrix with entries

$$(3.8) \quad \begin{aligned} a_{ij} = & |B| \delta_{ij} - \int_{\partial B} \Phi_i \frac{\partial \Phi_j}{\partial \nu} ds \\ & + \delta_n \left(\int_{\partial B} \frac{\partial \phi}{\partial \nu} ds \right)^{-1} \left(\int_{\partial B} \frac{\partial \Phi_i}{\partial \nu} ds \right) \left(\int_{\partial B} \frac{\partial \Phi_j}{\partial \nu} ds \right), \end{aligned}$$

where ν is the outward normal relative to B and $\delta_n = 0$ if $n = 2$, $\delta_n = 1$ if $n \geq 3$.

Proof. The fact that A is symmetric follows from (3.3). To verify that A is positive definite we compute

$$\begin{aligned} \sum a_{ij} \xi_i \xi_j = & |B| |\xi|^2 - \int_{\partial B} \chi \frac{\partial \chi}{\partial \nu} ds \\ & + \delta_n \left(\int_{\partial B} \frac{\partial \phi}{\partial \nu} ds \right)^{-1} \left(\int_{\partial B} \frac{\partial \chi}{\partial \nu} ds \right)^2 \end{aligned}$$

with $\chi = \sum \xi_i \Phi_i$, and note that

$$\begin{aligned} - \int_{\partial B} \chi \frac{\partial \chi}{\partial \nu} ds &= \int_{\mathbf{R}^2 \setminus B} |\nabla \chi|^2 dy > 0 \quad (n = 2), \text{ and} \\ - \int_{\partial B} \chi \frac{\partial \chi}{\partial \nu} ds &+ \left(\int_{\partial B} \frac{\partial \chi}{\partial \nu} ds \right)^2 \left(\int_{\partial B} \frac{\partial \phi}{\partial \nu} ds \right)^{-1} \\ &= \int_{\mathbf{R}^n \setminus B} |\nabla \chi|^2 dy - \left(\int_{\mathbf{R}^n \setminus B} \nabla \chi \cdot \nabla \phi dy \right)^2 \left(\int_{\mathbf{R}^n \setminus B} |\nabla \phi|^2 dy \right)^{-1} \geq 0 \quad (n \geq 3), \end{aligned}$$

through integration by parts and use of Lemma 3.1.

To prove (3.6) and (3.7) consider first the case in which there is only one inhomogeneity, with $z = 0$ and $\rho = 1 : \omega_\epsilon = \epsilon B$. Let d be a fixed positive number (chosen sufficiently small). Without loss of generality we may assume that

$$(3.9) \quad \epsilon B \subset \{|x| \leq \frac{d}{2}\} \subset \{|x| \leq 2d\} \subset \Omega.$$

Let χ_ϵ be the solution to

$$\begin{aligned} \nabla \cdot (\gamma \nabla \chi_\epsilon) &= 0 \quad \text{in } \{|x| < d\} \setminus \overline{\epsilon B}, \\ \chi_\epsilon &= 1 \quad \text{on } \partial(\epsilon B) \quad \text{and} \quad \chi_\epsilon = 0 \quad \text{on } \{|x| = d\}. \end{aligned}$$

Set

$$c_\epsilon = u_\epsilon|_{\epsilon B} \quad (\text{a constant}),$$

and introduce the functions

$$(3.10) \quad V_\epsilon(x) = u_\epsilon(x) - \chi_\epsilon(x)(c_\epsilon - U(0)), \quad x \in \{|x| \leq d\} \setminus \epsilon B,$$

and

$$W_\epsilon(y) = \frac{V_\epsilon(\epsilon y) - U(\epsilon y)}{\epsilon}, \quad y \in \{|y| \leq \frac{d}{\epsilon}\} \setminus B.$$

On ∂B :

$$(3.11) \quad \begin{aligned} W_\epsilon(y) &= \frac{U(0) - U(\epsilon y)}{\epsilon} = -\nabla U(0) \cdot y - \frac{\epsilon}{2} \sum D_{ij} U(\tilde{y}) y_i y_j \\ &\rightarrow -\nabla U(0) \cdot y \quad \text{as } \epsilon \rightarrow 0, \end{aligned}$$

uniformly in $C^{1+\beta'}$, for any $0 < \beta' < \beta$. On $|y| = d/\epsilon$:

$$W_\epsilon(y) = \frac{u_\epsilon(\epsilon y) - U(\epsilon y)}{\epsilon} = \frac{u_\epsilon(x) - U(x)}{\epsilon},$$

with $x = \epsilon y$, $|x| = d$. By Lemma 2.1

$$\int_{\{\frac{d}{2} \leq |x| \leq 2d\}} (|\nabla \frac{u_\epsilon - U}{\epsilon}|^2 + |\frac{u_\epsilon - U}{\epsilon}|^2) dx \leq C \epsilon^{n-2}.$$

Since

$$\nabla \cdot (\gamma \nabla \frac{u_\epsilon - U}{\epsilon}) = 0 \quad , \quad \text{in } \Omega \setminus \{|x| \leq d/2\},$$

we can apply elliptic estimates to deduce that

$$(3.12) \quad \max_{|x|=d} \left| \frac{u_\epsilon(x) - U(x)}{\epsilon} \right| \leq C \epsilon^{\frac{n-2}{2}}.$$

Therefore

$$(3.13) \quad |W_\epsilon(y)| \leq C \epsilon^{\frac{n-2}{2}} \quad \text{on } |y| = \frac{d}{\epsilon},$$

and at the same time W_ϵ solves

$$(3.14) \quad \nabla \cdot (\gamma(\epsilon y) \nabla W_\epsilon(y)) = 0 \quad \text{in } \{|y| < \frac{d}{\epsilon}\} \setminus \overline{B}.$$

The result in this lemma differs slightly depending on whether $n = 2$ or $n \geq 3$. We turn first to the case $n = 2$. From (3.11), (3.13) and (3.14) we obtain, using the maximum principle, that $W_\epsilon(y)$ is bounded in $\{|y| \leq R\} \setminus B$ uniformly in R and ϵ , provided $R < d/\epsilon$. By elliptic estimates and a compactness argument we deduce that $W_\epsilon \rightarrow W_0$ as $\epsilon \rightarrow 0$, where $W_0(y)$ is the solution of

$$\begin{aligned}\Delta W_0 &= 0 \quad \text{in } \mathbf{R}^2 \setminus B, \\ W_0 &= -\nabla U(0) \cdot y \quad \text{on } \partial B, \\ W_0 &\text{ is bounded in } \mathbf{R}^2 \setminus B;\end{aligned}$$

the convergence is in $C^{3+\beta'}(\{|y| \leq R\} \setminus B)$ for any $0 < \beta' < \beta$. In particular

$$(3.15) \quad \frac{\partial W_\epsilon}{\partial \nu} \rightarrow \frac{\partial W_0}{\partial \nu} \equiv m_0 \quad \text{uniformly on } \partial B.$$

Recalling the definition of W_ϵ we get

$$(3.16) \quad \frac{\partial V_\epsilon}{\partial \nu}(x) - \frac{\partial U}{\partial \nu}\left(\frac{x}{\epsilon}\right) - m_0\left(\frac{x}{\epsilon}\right) \rightarrow 0 \quad \text{on } \partial(\epsilon B);$$

here ν has been used simultaneously to denote a smooth unit normal field $\nu(x)$ on $\partial(\epsilon B)$ as well as its counterpart, $\nu(\epsilon y)$, on ∂B . From (3.16) we immediately get, upon inserting (3.10),

$$(3.17) \quad \frac{\partial u_\epsilon}{\partial \nu}(x) - \frac{\partial \chi_\epsilon}{\partial \nu}(x)(c_\epsilon - U(0)) - \nabla U(0) \cdot \nu(x) - m_0\left(\frac{x}{\epsilon}\right) \rightarrow 0$$

uniformly on $\partial(\epsilon B)$ as $\epsilon \rightarrow 0$. It is easy to see that

$$\int_{\partial(\epsilon B)} \gamma \frac{\partial u_\epsilon}{\partial \nu} ds = 0,$$

and (3.17) therefore yields

$$(3.18) \quad \frac{c_\epsilon - U(0)}{\epsilon} \int_{\partial(\epsilon B)} \gamma \frac{\partial \chi_\epsilon}{\partial \nu} ds + \gamma(0) \int_{\partial B} m_0(y) ds \rightarrow 0$$

as $\epsilon \rightarrow 0$. But

$$\int_{\partial B} m_0(y) ds = \lim_{R \rightarrow \infty} \int_{\{|y|=R\}} \frac{\partial W_0}{\partial \nu} ds = 0$$

as a consequence of the second identity in Lemma 3.1, so (3.18) gives

$$(3.19) \quad \frac{c_\epsilon - U(0)}{\epsilon} \int_{\partial(\epsilon B)} \frac{\partial \chi_\epsilon}{\partial \nu} ds \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.$$

The maximum principle shows that $\partial\chi_\epsilon/\partial\nu$ has constant sign on $\partial(\epsilon B)$, i.e., (3.19) may be written

$$(3.20) \quad \frac{|c_\epsilon - U(0)|}{\epsilon} \int_{\partial(\epsilon B)} \left| \frac{\partial\chi_\epsilon}{\partial\nu} \right| ds \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.$$

In conjunction with (3.17) this gives

$$\frac{1}{\epsilon} \int_{\partial(\epsilon B)} \left| \frac{\partial u_\epsilon}{\partial\nu} \right| ds \leq C,$$

which is exactly the assertion (3.6).

In terms of the solutions Φ_j of (3.1),

$$W_0 = \sum U_{x_j}(0) \Phi_j,$$

and thus

$$(3.21) \quad m_0 = \sum U_{x_j}(0) \frac{\partial}{\partial\nu} \Phi_j \quad \text{on } \partial B.$$

A combination of (3.17), (3.20) and (3.21) yields

$$(3.22) \quad \frac{1}{\epsilon^2} \int_{\partial(\epsilon B)} \gamma \frac{\partial u_\epsilon}{\partial\nu} x ds_x \rightarrow \gamma(0) U_{x_j}(0) \int_{\partial B} (\nu_j y_i + \frac{\partial\Phi_j}{\partial\nu} y_i) ds_y = \gamma(0) A \nabla U(0),$$

where $A = (a_{ij})$ is the matrix with

$$\begin{aligned} a_{ij} &= \int_{\partial B} (\nu_j y_i + \frac{\partial\Phi_j}{\partial\nu} y_i) ds_y \\ &= |B| \delta_{ij} - \int_{\partial B} \Phi_i \frac{\partial\Phi_j}{\partial\nu} ds, \end{aligned}$$

and (3.7), (3.8) follows.

We now turn to the case $n \geq 3$. The function

$$m(y) = \tilde{m}(r) = r^{\frac{2-n}{2}}, \quad r = |y|$$

satisfies

$$\begin{aligned} \nabla \cdot (\gamma(\epsilon y) \nabla m(y)) &\leq \gamma(\epsilon y) (\tilde{m}'' + \frac{n-1}{r} \tilde{m}') - (\|\nabla\gamma\|_{L^\infty}) d \frac{\tilde{m}'}{r} \\ &< 0 \quad \text{if} \quad |y| \leq \frac{d}{\epsilon}, \end{aligned}$$

provided d is sufficiently small. Using (3.11), (3.12) and (3.14), we may compare W_ϵ with $Cm(y)$ for some positive constant C (independent of ϵ) and conclude that

$$(3.23) \quad |W_\epsilon(y)| \leq C |y|^{\frac{2-n}{2}} \quad \text{for } y \in \{|y| \leq \frac{d}{\epsilon}\} \setminus B.$$

Therefore W_ϵ converges uniformly on compact sets towards W_0 , the unique solution of

$$(3.24) \quad \begin{aligned} \Delta W_0 &= 0 \quad \text{in } \mathbb{R}^n \setminus \overline{B}, \\ W_0 &= -\nabla U(0) \cdot y \quad \text{on } \partial B, \\ W_0(y) &\rightarrow 0 \quad \text{as } |y| \rightarrow \infty. \end{aligned}$$

The statements (3.15), (3.16) and (3.17) remain valid and, as before,

$$(3.25) \quad \begin{aligned} \frac{c_\epsilon - U(0)}{\epsilon^{n-1}} \int_{\partial(\epsilon B)} \gamma \frac{\partial \chi_\epsilon}{\partial \nu} ds &\rightarrow -\gamma(0) \int_{\partial B} m_0(y) ds \\ &= -\gamma(0) \sum U_{x_j}(0) \int_{\partial B} \frac{\partial \Phi_j}{\partial \nu} ds; \end{aligned}$$

only now the right-hand side is in general $\neq 0$. (If $\Phi_j = c_j |y|^{2-n} + O(|y|^{1-n})$ as $|y| \rightarrow \infty$, $n \geq 3$, then $\int_{\partial B} (\partial \Phi_j)/\partial \nu = 0$ if and only if $c_j = 0$). The function $\partial \chi_\epsilon/\partial \nu$ has a fixed sign on $\partial(\epsilon B)$, and by combining (3.25) with (3.17) we deduce that the assertion (3.6) holds.

From (3.17) we obtain a relation similar to (3.22) with the additional term

$$J \equiv \lim_{\epsilon \rightarrow 0} \frac{c_\epsilon - U(0)}{\epsilon^n} \int_{\partial(\epsilon B)} \gamma(x) \frac{\partial \chi_\epsilon}{\partial \nu} x ds_x$$

appearing on the right-hand side. Using the barrier $Cm(y)$ we see that the functions $\chi_\epsilon(\epsilon y)$ are all majorized by $Cm(y)$ on $\{|y| \leq \frac{d}{\epsilon}\} \setminus B$. Thus $\chi_\epsilon(\epsilon y) \rightarrow \phi(y)$ uniformly with its first derivatives in $\{|y| \leq R\} \setminus B$, for any $R > 0$. Combining this fact with (3.25), we get

$$J = -\gamma(0) \left(\sum U_{x_j}(0) \int_{\partial B} \frac{\partial \Phi_j}{\partial \nu} ds \right) \int_{\partial B} y \frac{\partial \phi}{\partial \nu} ds_y \left(\int_{\partial B} \frac{\partial \phi}{\partial \nu} ds \right)^{-1}.$$

The formula (3.8) follows by use of (3.4) in the above expression for J .

We have thus completed the proof of Lemma 3.2 in case of one inhomogeneity, with $z = 0$, $\rho = 1$; a simple rescaling and a translation gives the result for the general case of one

inhomogeneity. If there are more than one inhomogeneity, then we can apply the previous argument to the inhomogeneities one at a time, provided we verify that

$$(3.26) \quad \int_{z_k + \epsilon \rho_k B} \gamma \frac{\partial u_\epsilon}{\partial \nu} ds = 0$$

for each $1 \leq k \leq K$. To prove (3.26) we return to the u_ϵ^α defined in the introduction; it is clear that (3.26) holds for all u_ϵ^α , and since $u_\epsilon = \lim_{\alpha \rightarrow \infty} u_\epsilon^\alpha$ (in $H^1(\Omega)$) the same identity holds for u_ϵ . \square

Let $N(\cdot, y)$ denote the *Neumann function* in Ω corresponding to γ , i.e., the solution of

$$\begin{aligned} -\nabla \cdot (\gamma \nabla N) &= \delta_y \quad \text{in } \Omega, \\ \gamma \frac{\partial N}{\partial \nu} &= -\frac{1}{|\partial \Omega|} \quad \text{on } \partial \Omega, \end{aligned}$$

with

$$\int_{\partial \Omega} N ds = 0;$$

ν denotes the outward normal to $\partial \Omega$. If $y \in \Omega \setminus \bar{\omega}_\epsilon$ then

$$(3.27) \quad \begin{aligned} u_\epsilon(y) &= - \int_{\Omega} u_\epsilon \nabla \cdot (\gamma \nabla N) dx = - \int_{\partial \Omega} u_\epsilon \gamma \frac{\partial N}{\partial \nu} ds \\ &\quad - \int_{\partial \omega_\epsilon} \gamma \frac{\partial u_\epsilon}{\partial \nu} N ds + \int_{\partial \Omega} \gamma \frac{\partial u_\epsilon}{\partial \nu} N ds. \end{aligned}$$

Since $\gamma(\partial N / \partial \nu)$ is constant on $\partial \Omega$ and $\int_{\partial \Omega} u_\epsilon ds = 0$, it follows that $\int_{\partial \Omega} u_\epsilon \gamma(\partial N / \partial \nu) ds = 0$. Combining this and the identity $\gamma(\partial u_\epsilon / \partial \nu) = \psi$ on $\partial \Omega$ with (3.27), we get

$$(3.28) \quad u_\epsilon(y) = - \int_{\partial \omega_\epsilon} \gamma \frac{\partial u_\epsilon}{\partial \nu} N ds + \int_{\partial \Omega} \psi N ds.$$

Note that ν denotes the outward normal relative to ω_ϵ . The first term on the right-hand side of (3.28) may be written as

$$(3.29) \quad \begin{aligned} & - \sum_{k=1}^K \int_{\partial(z_k + \epsilon \rho_k B)} \gamma \frac{\partial u_\epsilon}{\partial \nu} N ds \\ &= - \sum_{k=1}^K N(z_k, y) \int_{\partial(z_k + \epsilon \rho_k B)} \gamma \frac{\partial u_\epsilon}{\partial \nu} ds - \nabla_z N(z_k, y) \cdot \int_{\partial(z_k + \epsilon \rho_k B)} \gamma \frac{\partial u_\epsilon}{\partial \nu} (x - z_k) ds_x \\ & \quad + O(\epsilon^2 \int_{\partial(z_k + \epsilon \rho_k B)} \gamma \left| \frac{\partial u_\epsilon}{\partial \nu} \right| ds). \end{aligned}$$

We already know that

$$\int_{\partial(z_k + \epsilon \rho_k B)} \gamma \frac{\partial u_\epsilon}{\partial \nu'} ds = 0.$$

Using Lemma 3.2 and (3.28), (3.29) we therefore get:

LEMMA 3.3. *There holds:*

$$(3.30) \quad \begin{aligned} u_\epsilon(y) = -\epsilon^n \sum_{k=1}^K \rho_k^n \gamma(z_k) \nabla_z N(z_k, y) \cdot A \nabla U(z_k) \\ + \int_{\partial\Omega} \psi(x) N(x, y) ds_x + \epsilon^n \eta(\epsilon, y, \{\rho_k\}, \{z_k\}), \end{aligned}$$

where the matrix $A = (a_{ij})$ is given by (3.8) and $\eta(\epsilon, y, \{\rho_k\}, \{z_k\})$ as well as $\nabla_y \eta(\epsilon, y, \{\rho_k\}, \{z_k\})$ converges to zero, as ϵ approaches zero, uniformly with respect to $y \in \bar{\Omega}$, $\{\rho_k\}$ and $\{z_k\}$, provided $\text{dist}(y, \{z_k\}) \geq d > 0$.

Note that

$$\int_{\partial\Omega} \psi(x) N(x, y) ds_x = U(y).$$

In general the matrix A cannot be computed explicitly. One exception is the case when B is the unit ball. In that case $\Phi_j = -y_j / |y|^n$ and (for $n \geq 3$) $\phi = |y|^{2-n}$. A simple computation gives $A = \omega_n I$, where I is the identity matrix and $\omega_n = \text{meas} \{|y| = 1\}$.

§4 Continuous dependence. The Neumann function, introduced earlier is symmetric in its arguments in

$(\Omega \times \Omega) \setminus \text{diag}(\Omega \times \Omega)$. It furthermore has the form

$$(4.1) \quad \begin{aligned} N(z, y) = N(y, z) &= -\frac{1}{2\pi\gamma(z)} \log |y - z| + R(y, z) & \text{if } n = 2 \\ &= \frac{1}{(n-2)\omega_n\gamma(z)} |y - z|^{2-n} + R(y, z) & \text{if } n \geq 3, \end{aligned}$$

where $R(y, z)$ solves

$$(4.2) \quad \begin{aligned} -\nabla_y \cdot (\gamma(y) \nabla_y R(y, z)) &= -\frac{1}{\omega_n \gamma(z)} \frac{\nabla \gamma(y) \cdot (y - z)}{|y - z|^n}, & y \in \Omega, \\ \gamma(y) \frac{\partial}{\partial \nu_y} R(y, z) &= -\frac{1}{|\partial\Omega|} + \frac{\gamma(y)}{\omega_n \gamma(z)} \frac{(y - z) \cdot \nu_y}{|y - z|^n}, & y \in \partial\Omega; \end{aligned}$$

R is not symmetric in its arguments. Since the function

$$y \rightarrow \frac{\nabla \gamma(y) \cdot (y - z)}{|y - z|^n}$$

is in $L^p(\Omega)$ for any $1 \leq p < \frac{n}{n-1}$, it follows from (4.2) and elliptic estimates that $R(\cdot, z)$ is in $W^{2,p}(\Omega)$ for any $z \in \Omega, 1 \leq p < \frac{n}{n-1}$. From (4.1) we thus get

$$(4.3) \quad \begin{aligned} N(z, \cdot) &\in W^{1,p}(\Omega) \text{ for } 1 \leq p < \frac{n}{n-1}, z \in \Omega, \\ N(z, \cdot) &\notin W^{2,1}(\Omega \setminus \{z\}) \text{ for } z \in \Omega. \end{aligned}$$

It is easy to verify that the function $R(y, z)$ is differentiable with respect to $z \in \Omega$. In fact, for any fixed vector $\alpha \neq 0$, the function $R_\alpha(\cdot, z) = \alpha \cdot \nabla_z R(\cdot, z)$ is the solution to

$$(4.4a) \quad -\nabla_y(\gamma(y)\nabla_y R_\alpha(y, z)) = -\frac{1}{\omega_n} \alpha \cdot \nabla_z \left(\frac{\nabla \gamma(y) \cdot (y-z)}{\gamma(z) |y-z|^n} \right), \quad y \in \Omega,$$

$$(4.4b) \quad \gamma(y) \frac{\partial}{\partial \nu_y} R_\alpha(y, z) = \frac{1}{\omega_n} \alpha \cdot \nabla_z \left(\frac{\gamma(y)(y-z) \cdot \nu_y}{\gamma(z) |y-z|^n} \right), \quad y \in \partial\Omega,$$

with the normalization

$$(4.5) \quad \begin{aligned} \int_{\partial\Omega} R_\alpha(y, z) ds_y &= \frac{1}{2\pi} \int_{\partial\Omega} \alpha \cdot \nabla_z \left(\frac{\log |y-z|}{\gamma(z)} \right) ds_y \quad \text{if } n=2, \\ &= -\frac{1}{(n-2)\omega_n} \int_{\partial\Omega} \alpha \cdot \nabla_z \left(\frac{|y-z|^{2-n}}{\gamma(z)} \right) ds_y \quad \text{if } n \geq 3. \end{aligned}$$

The right hand side of (4.4a)

$$\begin{aligned} -\frac{1}{2\pi} \alpha \cdot \nabla_z \left(\frac{\nabla \gamma(y) \cdot (y-z)}{\gamma(z) |y-z|^2} \right) &= -\frac{1}{2\pi} \nabla \gamma(y) \cdot \nabla_y \left(\alpha \cdot \nabla_z \frac{\log |y-z|}{\gamma(z)} \right) \quad (n=2), \\ -\frac{1}{\omega_n} \alpha \cdot \nabla_z \left(\frac{\nabla \gamma(y) \cdot (y-z)}{\gamma(z) |y-z|^n} \right) &= \frac{1}{(n-2)\omega_n} \nabla \gamma(y) \cdot \nabla_y \left(\alpha \cdot \nabla_z \frac{1}{\gamma(z) |y-z|^{n-2}} \right) \quad (n \geq 3), \end{aligned}$$

is in $(W^{1,q}(\Omega))'$, $n < q$, since both

$$\nabla_z \frac{\log |y-z|}{\gamma(z)} \quad (n=2) \quad \text{and} \quad \nabla_z \frac{1}{\gamma(z) |y-z|^{n-2}} \quad (n \geq 3)$$

are in $L^p(\Omega)$, $1 \leq p < \frac{n}{n-1}$, and are continuous on $\partial\Omega$ ($z \in \Omega$). The right hand side of (4.4b) is continuous on $\partial\Omega$ ($z \in \Omega$). By elliptic regularity and duality it now follows that

$$(4.6) \quad R_\alpha(\cdot, z) \in W^{1,p}(\Omega) \text{ for } 1 \leq p < \frac{n}{n-1}, z \in \Omega.$$

The functions

$$\alpha \cdot \nabla_z (\log |y-z|) = \frac{(z-y) \cdot \alpha}{|y-z|^2} \quad (n=2)$$

and

$$\alpha \cdot \nabla_z (|y - z|^{2-n}) = (2-n) \frac{(z-y) \cdot \alpha}{|y-z|^n} \quad (n \geq 3)$$

are in $L^p(\Omega)$ but not in $W^{1,1}(\Omega \setminus \{z\})$. Consequently, it follows from (4.1) and (4.6) that

$$(4.7) \quad \begin{aligned} \alpha \cdot \nabla_z N(z, \cdot) &\in L^p(\Omega) \text{ for } 1 \leq p < \frac{n}{n-1}, z \in \Omega, \text{ and} \\ \alpha \cdot \nabla_z N(z, \cdot) &\notin W^{1,1}(\Omega \setminus \{z\}) \text{ for } z \in \Omega \end{aligned}$$

for any vector $\alpha \neq 0$.

Proceeding once more by the method used to prove (4.6) one can show that

$$(4.8) \quad D_z^2 R(\cdot, z) \in L^p(\Omega) \quad \text{for } 1 \leq p < \frac{n}{n-1}, z \in \Omega.$$

Differentiation of $\log |y - z|$ gives

$$(D_z^2 \log |y - z| \alpha) \cdot \beta = \frac{\alpha \cdot \beta}{|y - z|^2} - \frac{2((z - y) \cdot \alpha)((z - y) \cdot \beta)}{|z - y|^4},$$

from which we conclude that, in the case $n = 2$,

$$(D_z^2 \log |y - z| \alpha) \cdot \beta \quad \text{is not in } L^1(\Omega \setminus \{z\})$$

for any vectors $\alpha \neq 0, \beta \neq 0$. The same statement holds in dimension $n \geq 3$ for the function $|y - z|^{2-n}$. A combination of this with (4.1), (4.8) yields

$$(4.9) \quad (D_z^2 N(z, \cdot) \alpha) \cdot \beta \quad \text{is not in } L^1(\Omega \setminus \{z\})$$

for any vectors $\alpha \neq 0, \beta \neq 0$.

Introduce the function F :

$$(4.10) \quad F(\{\rho_k\}_1^K, \{z_k\}_1^K)(y) = \sum_{k=1}^K \rho_k^n \gamma(z_k) \nabla_z N(z_k, y) \cdot A \nabla U(z_k).$$

The Frechet derivative of F with respect to $\{\rho_k\}_1^K, \{z_k\}_1^K$ is the linear expression

$$(4.11) \quad \begin{aligned} &DF(\{\Delta \rho_k\}_1^K, \{\Delta z_k\}_1^K)(y) \\ &= \sum_{k=1}^K [\rho_k^n \gamma(z_k) (D_z^2 N(z_k, y) A \nabla U(z_k)) \cdot \Delta z_k \\ &\quad + \rho_k^n (\nabla \gamma(z_k) \cdot \Delta z_k) (\nabla_z N(z_k, y) \cdot A \nabla U(z_k)) \\ &\quad + \rho_k^n \gamma(z_k) \nabla_z N(z_k, y) \cdot (A D_z^2 U(z_k) \Delta z_k) \\ &\quad + n \rho_k^{n-1} (\Delta \rho_k) \gamma(z_k) \nabla_z N(z_k, y) \cdot A \nabla U(z_k)], \quad y \in \overline{\Omega} \setminus \{z_k\}_1^K. \end{aligned}$$

LEMMA 4.1. If

$$\sum (|\Delta \rho_k| + |\Delta z_k|) \neq 0, \rho_k > 0 \text{ and } \nabla U(z_k) \neq 0,$$

then the function $DF(\{\Delta \rho_k\}_1^K, \{\Delta z_k\}_1^K)(\cdot)$ is not identically zero in $\Omega \setminus \{z_k\}_1^K$.

Proof. If at least one Δz_k is not zero, then it follows from (4.9), (4.11) and (4.7) (with $p = 1$) that DF is not in $L^1(\Omega \setminus \{z_k\}_1^K)$ and thus not identically zero; here we used the fact that A is positive definite and $\nabla U(z_k) \neq 0$. If all the Δz_k equal zero then at least one $\Delta \rho_k$ is not equal to zero, and DF is not in $W^{1,1}(\Omega \setminus \{z_k\}_1^K)$, by (4.7); thus again $DF \neq 0$. \square

For the following lemma we need all the assumptions made earlier in the introduction, in particular (1.3), (1.4) and (1.15). Let

$$\begin{aligned} & H(\{\rho_k\}_1^K, \{z_k\}_1^K; \{\rho'_k\}_1^{K'}, \{z'_k\}_1^{K'}) \\ &= \|F(\{\rho_k\}, \{z_k\}) - F(\{\rho'_k\}, \{z'_k\})\|_{L^\infty(\Gamma)} \\ &+ \left\| \frac{\partial}{\partial \nu} [F(\{\rho_k\}, \{z_k\}) - F(\{\rho'_k\}, \{z'_k\})] \right\|_{L^\infty(\Gamma)}, \end{aligned}$$

where Γ is a fixed nonempty, open subset of $\partial\Omega$.

LEMMA 4.2. There exists a positive constant δ such that if

$$H(\{\rho_k\}_1^K, \{z_k\}_1^K; \{\rho'_k\}_1^{K'}, \{z'_k\}_1^{K'}) < \delta$$

then

(i) $K = K'$ and, after appropriate reordering,

(ii) $|z_k - z'_k| + |\rho_k - \rho'_k| \leq CH(\{\rho_k\}_1^K, \{z_k\}_1^K, \{\rho'_k\}_1^K, \{z'_k\}_1^K), 1 \leq k \leq K$.

The constants δ and C depend on the same parameters as δ_0, ϵ_0 and C in Theorem 1.1.

Proof. Suppose the assertion $K = K'$ is not true. Then there exist sequences

$$\begin{aligned} \underline{z}^{(m)} &= \{z_k^{(m)}\}_1^K, \underline{\rho}^{(m)} = \{\rho_k^{(m)}\}_1^K, \\ \underline{z}^{(m)'} &= \{z_k^{(m)'}\}_1^{K'}, \underline{\rho}^{(m)'} = \{\rho_k^{(m)'}\}_1^{K'} \end{aligned}$$

with $K \neq K'$ such that the corresponding H converges to 0 as $m \rightarrow \infty$ (since $K, K' \leq M_0 = |\Omega| / |\{x \mid |x| \leq d_0/2\}|$, we can choose K and K' to be independent of m). Passing to a subsequence we get

$$\underline{z}^{(m)} \rightarrow \underline{z}, \underline{z}^{(m)'} \rightarrow \underline{z}', \underline{\rho}^{(m)} \rightarrow \underline{\rho} \text{ and } \underline{\rho}^{(m)'} \rightarrow \underline{\rho}'.$$

Since H approaches zero,

$$\begin{aligned} F(\underline{\rho}, \underline{z})(y) &= \lim_{m \rightarrow \infty} F(\underline{\rho}^{(m)}, \underline{z}^{(m)})(y) \\ &= \lim_{m \rightarrow \infty} F(\underline{\rho}^{(m)'} , \underline{z}^{(m)'})(y) = F(\underline{\rho}', \underline{z}')(y), \quad y \in \Gamma, \end{aligned}$$

and similarly for $\frac{\partial}{\partial \nu} F$

$$\frac{\partial}{\partial \nu} F(\underline{\rho}, \underline{z})(y) = \frac{\partial}{\partial \nu} F(\underline{\rho}', \underline{z}')(y), \quad y \in \Gamma.$$

Thus the function $G(y) \equiv F(\underline{\rho}, \underline{z})(y) - F(\underline{\rho}', \underline{z}')(y)$ is a solution of

$$\nabla \cdot (\gamma(y) \nabla G) = 0 \text{ in } \Omega \setminus (\{z_k\}_1^K \cup \{z'_k\}_1^{K'})$$

which vanishes together with its first derivatives on Γ . By uniqueness of the solution to the Cauchy problem ([3], [7]) it follows that G is identically zero, i.e.,

$$(4.12) \quad F(\underline{\rho}, \underline{z}) = F(\underline{\rho}', \underline{z}') \text{ in } \Omega \setminus (\{z_k\}_1^K \cup \{z'_k\}_1^{K'}).$$

The components of $\underline{\rho}$ and $\underline{\rho}'$ are all positive, so $F(\underline{\rho}, \underline{z})$ has a singularity at precisely each of the points z_k , and $F(\underline{\rho}', \underline{z}')$ has a singularity at precisely each of the points z'_k (by (4.7)). Therefore (4.12) is a contradiction, and we conclude that $K = K'$. The same argument shows that, after reordering,

$$(4.13) \quad |z_k - z'_k| + |\rho_k - \rho'_k| \rightarrow 0 \quad \text{as } H \rightarrow 0.$$

The proof of (ii) proceeds along the same lines. Assume that the assertion is not true. Then there exist sequences $\underline{z}^{(m)}, \underline{\rho}^{(m)}, \underline{z}^{(m)'} \text{ and } \underline{\rho}^{(m)'}$ (each element a K -vector) such that

$$(4.14) \quad \frac{H(\underline{\rho}^{(m)}, \underline{z}^{(m)}; \underline{\rho}^{(m)'}, \underline{z}^{(m)'})}{\sum_1^K (|z_k^{(m)} - z_k^{(m)'}| + |\rho_k^{(m)} - \rho_k^{(m)'}|)} \rightarrow 0.$$

Since the denominator in (4.14) is bounded, it follows that $H \rightarrow 0$, and thus by (4.13)

$$d^{(m)} = \sum_1^K (|z_k^{(m)} - z_k^{(m)'}| + |\rho_k^{(m)} - \rho_k^{(m)'}|) \rightarrow 0,$$

after reordering. Passing to a subsequence we may assume that

$$\begin{aligned} \lim \underline{z}^{(m)} &= \lim \underline{z}^{(m)'} = \underline{z}, \\ \lim \underline{\rho}^{(m)} &= \lim \underline{\rho}^{(m)'} = \underline{\rho}, \\ \lim \frac{z_k^{(m)} - z_k^{(m)'}}{d^{(m)}} &= \Delta z_k, \\ \lim \frac{\rho_k^{(m)} - \rho_k^{(m)'}}{d^{(m)}} &= \Delta \rho_k \quad ; \end{aligned}$$

note that

$$\sum_1^K (|\Delta z_k| + |\Delta \rho_k|) = 1.$$

From (4.14) and the definition of H it follows that

$$\begin{aligned} & \|DF(\{\Delta \rho_k\}_1^K, \{\Delta z_k\}_1^K)\|_{L^\infty(\Gamma)} \\ & + \left\| \frac{\partial}{\partial \nu} DF(\{\Delta \rho_k\}_1^K, \{\Delta z_k\}_1^K) \right\|_{L^\infty(\Gamma)} = 0, \end{aligned}$$

where DF denotes the Frechet derivative of F at $(\underline{\rho}, \underline{z})$. However, $DF(\{\Delta \rho_k\}, \{\Delta z_k\})(y)$ is clearly a solution of the homogeneous equation

$$\nabla_y \cdot (\gamma(y) \nabla_y (DF)(y)) = 0 \quad \text{in } \Omega \setminus \{z_k\}_1^K$$

(see (4.11)), and by uniqueness of the solution to the Cauchy problem

$$DF(\{\Delta \rho_k\}_1^K, \{\Delta z_k\}_1^K)(y) = 0 \quad \text{in } \Omega \setminus \{z_k\}_1^K.$$

This is a contradiction to Lemma 4.1. \square

Proof of Theorem 1.1. If $\|u_\epsilon - u'_\epsilon\|_{L^\infty(\Gamma)} < \delta_0 \epsilon^n$ then the representation formula in Lemma 3.3 gives

$$\begin{aligned} \|F(\{\rho_k\}_1^K, \{z_k\}_1^K) - F(\{\rho'_k\}_1^{K'}, \{z'_k\}_1^{K'})\|_{L^\infty(\Gamma)} & \leq \epsilon^{-n} \|u_\epsilon - u'_\epsilon\|_{L^\infty(\Gamma)} + \tilde{\eta}(\epsilon) \\ & < \delta_0 + \tilde{\eta}(\epsilon), \end{aligned}$$

and since $\frac{\partial u_\epsilon}{\partial \nu} = \frac{\partial u'_\epsilon}{\partial \nu}$ on Γ it also follows that

$$\left\| \frac{\partial}{\partial \nu} F(\{\rho_k\}_1^K, \{z_k\}_1^K) - \frac{\partial}{\partial \nu} F(\{\rho'_k\}_1^{K'}, \{z'_k\}_1^{K'}) \right\|_{L^\infty(\Gamma)} \leq \tilde{\eta}(\epsilon);$$

here $\eta(\epsilon)$ denotes the maximum of $2(|\eta(\epsilon, y, \{\rho_k\}, \{z_k\})| + |\nabla_y \eta(\epsilon, y, \{\rho_k\}, \{z_k\})|)$ (from Lemma 3.3) over $y \in \Gamma, \{\rho_k\} \leq [d_0, D_0]$ and $\{z_k\} \subseteq \Omega$, subject to (1.3). In terms of H ,

$$H(\{\rho_k\}_1^K, \{z_k\}_1^K; \{\rho'_k\}_1^{K'}, \{z'_k\}_1^{K'}) < \delta_0 + 2\tilde{\eta}(\epsilon) < \delta$$

if we take $\delta_0 = \delta/2$ and $\epsilon < \epsilon_0$ so that $2\tilde{\eta}(\epsilon) < \delta/2$. Applying Lemma 4.2 we conclude that $K = K'$ and

$$|z_k - z'_k| + |\rho_k - \rho'_k| \leq CH \leq C\epsilon^{-n} \|u - u'_\epsilon\|_{L^\infty(\Gamma)} + \eta(\epsilon),$$

where $\eta(\epsilon) = 2C\tilde{\eta}(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$.

REMARK 4.1. If $n = 2$ and if $U|_{\partial\Omega}$ has only finitely many relative extrema then there are at most finitely many points in Ω where ∇U can vanish (cf. [1]). If the points z_1, \dots, z_K are restricted to stay away from these critical points, then the non-degeneracy condition (1.15) may be dropped.

REMARK 4.2. In the special case of a constant reference conductivity γ and $n = 2$, the results in this section have some similarity to the results about the location of poles for meromorphic functions, found in [16]. If Γ is all of $\partial\Omega$ then Lemma 4.2 follows directly from the representation formula in Section 3 and the analysis in [16]. On the other hand if Γ is a proper subset of $\partial\Omega$ then this analysis would give a Hölder estimate and not the stronger Lipschitz estimate of Lemma 4.2.

REMARK 4.3. One could raise the question of "best possible" choice for Γ or ψ , or what are the shapes B that are the easiest or most difficult to find. This analysis would be somewhat in the spirit of [10], although the results would differ due to the different objectives.

§5. Small inhomogeneities with zero conductivity. The results of Sections 1–4 can be extended to the case where the inhomogeneities have zero conductivity, that is, to the case where u_ϵ^∞ is replaced by u_ϵ^0 . For $n = 2$ this follows from the fact that problems (1.5) and (1.6) are very closely related by duality. The flux $\sigma_\epsilon = \gamma \nabla u_\epsilon^0$ in $\Omega \setminus \omega_\epsilon$, $\sigma_\epsilon = 0$ in ω_ϵ , minimizes the functional

$$\frac{1}{2} \int_{\Omega} \gamma^{-1} |\sigma|^2 dx$$

subject to $\nabla \cdot \sigma = 0$ in Ω , $\sigma \cdot \nu = \psi$ on $\partial\Omega$ and $\sigma = 0$ in ω_ϵ . Assuming that Ω is simply connected we get that

$$\sigma_\epsilon = \left(-\frac{\partial}{\partial x_2} v_\epsilon, \frac{\partial}{\partial x_1} v_\epsilon \right),$$

where v_ϵ is the minimizer of

$$\frac{1}{2} \int_{\Omega} \gamma^{-1} |\nabla v|^2 dx$$

subject to $\nabla v = 0$ in ω_ϵ and $v = \Psi$ on $\partial\Omega$ (Ψ is determined as a clockwise integral of ψ). The Neumann problem (1.6) with inhomogeneities of conductivity 0 is therefore equivalent to a Dirichlet problem with infinitely conducting inhomogeneities and reference conductivity γ^{-1} . Since the method of §§1–4 applies equally well to the case of the Dirichlet problem, we conclude that Theorem 1.1 extends to problem (1.6), i.e., to $u_\epsilon = u_\epsilon^0$.

Consider now the case $n \geq 3$. The results of Section 2 can easily be extended to $u_\epsilon = u_\epsilon^0$. To derive a representation formula for $u_\epsilon = u_\epsilon^0$, we start with

$$(5.1) \quad u_\epsilon(y) = \int_{\partial\omega_\epsilon} \gamma u_\epsilon \frac{\partial N}{\partial \nu} ds + \int_{\partial\Omega} \psi N ds$$

(cf. (3.27)) where ν is the outward normal relative to ω_ϵ . Suppose there is only one inhomogeneity, with $z = 0, \rho = 1$. We assume that B is star-shaped with respect to the origin and

$$(5.2) \quad \frac{\partial r}{\partial \nu} \neq 0 \text{ along } \partial B \quad (r = |y|).$$

Let $\Psi'_i(y)$ and $\Psi_i(y)$ denote the unique solutions to

$$(5.3) \quad \begin{aligned} \nabla \cdot (\gamma(\epsilon y) \nabla \Psi'_i(y)) &= 0 \text{ in } \{|y| < \frac{d}{\epsilon}\} \setminus \overline{B}, \\ \frac{\partial \Psi'_i}{\partial \nu} &= -\nu_i \text{ on } \partial B, \\ \Psi'_i(y) &= 0 \text{ on } \{|y| = \frac{d}{\epsilon}\} \end{aligned}$$

and

$$(5.4) \quad \begin{aligned} \Delta \Psi_i &= 0 \text{ in } \mathbf{R}^n \setminus \overline{B}, \\ \frac{\partial \Psi_i}{\partial \nu} &= -\nu_i \text{ on } \partial B, \\ \Psi_i(y) &\rightarrow 0 \text{ as } |y| \rightarrow \infty, \end{aligned}$$

where ν is the outward normal relative to B . Notice that the function $m(y) = \tilde{m}(r) = r^{\frac{2-n}{2}}$, $r = |y|$, satisfies

$$\frac{\partial m}{\partial \nu} < 0 \text{ on } \partial B \quad (\text{by (5.2)}),$$

and therefore $Cm(y)$ can be used as a barrier for the Ψ'_i . We conclude that $\Psi'_i \rightarrow \Psi_i$ in C^1 in any set $\{|y| \leq R\} \setminus B$ as $\epsilon \rightarrow 0$.

Consider the function

$$W_\epsilon(y) = \frac{u_\epsilon(\epsilon y) - U(\epsilon y)}{\epsilon} - \sum U_{x_i}(0) \Psi'_i(y).$$

From energy estimates it follows, as in section 3, that

$$\begin{aligned} |W_\epsilon(y)| &\leq C |y|^{\frac{2-n}{2}} \\ &= C \frac{\epsilon^\theta}{d^\theta} |y|^{\frac{2-n}{2} + \theta} \leq \delta |y|^{\frac{2-n}{2} + \theta} \text{ on } |y| = \frac{d}{\epsilon}, \end{aligned}$$

provided $\epsilon^\theta < d^\theta \delta / C$, $0 < \theta < 1/2$. At the same time

$$\frac{\partial W_\epsilon}{\partial \nu} \rightarrow 0 \text{ uniformly on } \partial B, \text{ as } \epsilon \rightarrow 0.$$

Therefore, by comparison,

$$|W_\epsilon(y)| \leq \delta |y|^{\frac{2-n}{2}} + \theta \text{ in } \{|y| \leq \frac{d}{\epsilon}\} \setminus B$$

for ϵ sufficiently small. Since δ is arbitrary, we conclude that

$$(5.5) \quad u_\epsilon(x) = U(x) + \epsilon \sum U_{x_i}(0) \Psi_i\left(\frac{x}{\epsilon}\right) + \epsilon \tilde{\eta}(\epsilon, x), \quad x \in \partial(\epsilon B),$$

where $\tilde{\eta}(\epsilon, x) \rightarrow 0$ as $\epsilon \rightarrow 0$, uniformly in $x \in \partial(\epsilon B)$. Inserting this into (5.1) we get

$$(5.6) \quad \begin{aligned} u_\epsilon(y) = & \int_{\partial(\epsilon B)} \gamma(x) U(x) \frac{\partial N(x, y)}{\partial \nu_x} ds_x \\ & + \epsilon \int_{\partial(\epsilon B)} \sum \gamma(x) U_{x_i}(0) \Psi_i\left(\frac{x}{\epsilon}\right) \frac{\partial N(x, y)}{\partial \nu_x} ds_x + \epsilon^n \eta(\epsilon, y) + \int_{\partial \Omega} \psi(x) N(x, y) ds_x \end{aligned}$$

where $\eta(\epsilon, y) \rightarrow 0$ uniformly in y , $y \in \overline{\Omega}$, $|y| \geq d > 0$; ν is the outward normal relative to ϵB .

The first integral on the right-hand side can be written as

$$\begin{aligned} I_1 &= \epsilon^{n-1} \int_{\partial B} \gamma(\epsilon z) U(\epsilon z) \left(\frac{\partial}{\partial \nu_x} N \right)(\epsilon z, y) ds_z \\ &= \epsilon^n \sum U_{x_i}(0) \int_{\partial B} \gamma(\epsilon z) z_i \left(\frac{\partial}{\partial \nu_x} N \right)(\epsilon z, y) ds_z + O(\epsilon^{n+1}), \end{aligned}$$

since $U(\epsilon z) = U(0) + \sum \epsilon z_i U_{x_i}(0) + O(\epsilon^2)$ and

$$\int_{\partial B} \gamma(\epsilon z) \frac{\partial}{\partial \nu_x} N(\epsilon z, y) ds_z = 0.$$

Thus

$$I_1 = \epsilon^n \sum \gamma(0) U_{x_i}(0) N_{x_j}(0, y) \int_{\partial B} z_i \nu_j ds_z + O(\epsilon^{n+1})$$

where ν_j is the j -th component of the outward normal ν

The second integral on the right-hand side of (5.6) can be written as

$$I_2 = \epsilon^n \sum \gamma(0) U_{x_i}(0) N_{x_j}(0, y) \int_{\partial B} \Psi_i(z) \nu_j ds_z + O(\epsilon^{n+1}).$$

In summary

$$u_\epsilon(y) = \epsilon^n \gamma(0) \nabla_x N(0, y) \cdot A \nabla U(0) \\ + \int_{\partial\Omega} \psi(x) N(x, y) ds_x + \epsilon^n \eta(\epsilon, y)$$

with $A = (a_{ij})$ given by

$$(5.7) \quad a_{ij} = \int_{\partial B} \nu_i(z_j + \Psi_j(z)) ds_z,$$

ν is the outward normal relative to B , and $\eta(\epsilon, y) \rightarrow 0$ as $\epsilon \rightarrow 0$, uniformly with respect to y , $|y| \geq d > 0$.

Similarly, for the case of K rescaled inhomogeneities, we get the representation formula

$$(5.8) \quad u_\epsilon(y) = \epsilon^n \sum_{k=1}^K \rho_k^n \gamma(z_k) \nabla_z N(z_k, y) \cdot A \nabla U(z_k) \\ + \int_{\partial\Omega} \psi(x) N(x, y) ds_x + \epsilon^n \eta(\epsilon, y, \{\rho_k\}\{z_k\});$$

the only difference between this and (3.30) is the sign in front of the first term and the matrix (a_{ij}) , which here is given by (5.7).

From the definition of Ψ_j , (5.4), and integration by parts (using Lemma 3.1) it follows that

$$\sum a_{ij} \xi_i \xi_j = |B| |\xi|^2 + \int_{\mathbb{R}^n \setminus B} |\nabla \Psi|^2 dz$$

with $\Psi = \sum \xi_j \Psi_j$; consequently A is symmetric positive definite. In general A cannot be computed explicitly, however in case B is the unit ball

$$\Psi_i(z) = -\frac{1}{1-n} \frac{z_i}{|z|^n}$$

and we compute that

$$a_{ij} = \delta_{ij} \frac{\omega_n}{(n-1)}, \quad \omega_n = \text{meas} \{|z|=1\}.$$

Our main result, Theorem 1.1., now immediately carries over to the case of u_ϵ^0 .

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